Solutions

6.3-6.4: Decision Algorithms, Combinations and Permutations

Question 1. At a restaurant you can choose among 8 chicken dishes, 10 beef dishes, 4 seafood dishes and 12 vegetarian dishes. How many total options does this give you?

$8+10+4+12=34$ options.

Addition Principle. When choosing among r disjoint alternatives, suppose that

alternative 1 has n_1 possible outcomes
alternative 2 has n_2 possible outcomes
...
alternative r has n_r possible outcomes

with no two of these outcomes the same. Then the total number of possible outcomes is $n_1 + n_2 + \cdots + n_r$.

Question 2. Now suppose you are at a restaurant and you can choose among 5 appetizers, 34 main dishes and 10 desserts. How many total meal options (including one appetizer, one main dish and one dessert) can you choose from?

$5 \cdot 34 \cdot 10 = 1{,}700$ different meals.

Multiplication Principle. When making a sequence of choices with r steps, suppose that

step 1 has n_1 possible outcomes
step 2 has n_2 possible outcomes
...
step r has n_r possible outcomes

and that each sequence of choices results in a distinct outcome. Then the total number of possible outcomes is $n_1 \cdot n_2 \cdot \cdots \cdot n_r$.

Decision Algorithms.

Example 1. At an ice cream parlor you can choose between 15 flavors of ice cream and 5 flavors of frozen yogurt. In addition, you can choose to add one of ten toppings to your ice cream or yogurt and you can choose among 3 different sizes of cones for your ice cream and 2 different sizes of cups for your yogurt. How many different deserts can you choose from?

Algorithm:

Alternative 1: Ice Cream

- Choose a flavor
- Choose a topping
- Choose a size

$15 \cdot 10 \cdot 3 = 450$ choices

Alternative 2: Yogurt

- Choose a flavor
- Choose a topping
- Choose a size

$5 \cdot 10 \cdot 2 = 100$ choices

So $450 + 100 = 550$ possible desserts

Tree:

Hard to draw with 550 outcomes.

Example 2. You are playing Scrabble and losing. In your hand you have the letters k, e, r and e to work with. You wish to play all four letters to get yourself back into a manageable position. You can think of any four-letter words with this combination of letters, so you decide to list all possibilities. How large is your list?

No alternatives:

Step 1: 4 choices for first letter

Step 2: 3 choices for second letter

Step 3: 2 choices for third letter

Step 4: 1 choice for fourth letter

Gives $4 \cdot 3 \cdot 2 \cdot 1 = 24$ choices. But what is wrong with algorithm

There are 2 e's!! So try again.

Step 1: 4 choices for placing k

Step 2: 3 choices for placing r

Step 3: Place e's in remaining slots. 1 choice!

Gives $4 \cdot 3 \cdot 1 = 12$ choices

Question 3. Your nasty English teacher (nasty compared to your awesome math teacher) is making five students give a speech in class. None of the five wish to go first so the teacher will have to choose the order they give their speeches. How many possible orderings of the five students can the teacher choose?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ possible orders.

$= 5!$ possible orders.

Definition 1. We call an ordered list of items a <u>permutation</u> of those items.

$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n$ factorial.

$0! = 1$. This allows us to write $n! = n \cdot (n-1)!$

Question 4. There are one hundred people running in a race. However, the podium can only hold three people (first, second and third). How many different ways can the podium be ordered?

Step 1: 100 options for first

Step 2: 99 options for second

Step 3: 98 options for third

$\frac{100!}{97!}$

$P(n,r) = {}_nP_r = \frac{n!}{(n-r)!}$

$100 \cdot 99 \cdot 98 = 970{,}200$ possible podium orderings

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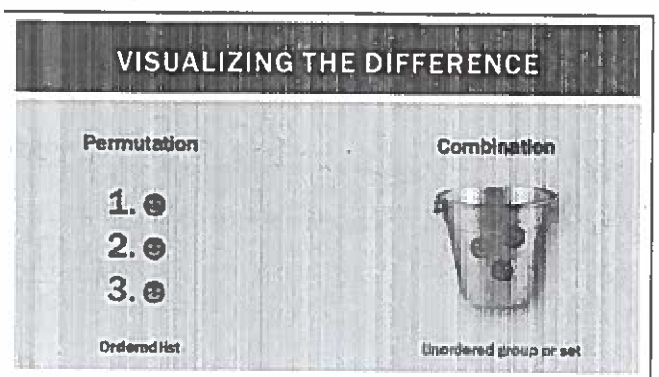
Permutations and Combinations. Definition 2. A <u>permutation</u> of n items taken r at a time is an *ordered* list of r items chosen from n. A <u>combination</u> of n items taken r at a time is an *unordered* set of r items chosen from n.



$C(n,r) = {}_nC_r = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

Question 5. Go back to question 4. There are still one hundred people in the race and still only three can stand on the podium. However, this time you are not worried about the order of the people on the podium, you are only worried about the collection of people on the podium. How many different combinations of people can there be?

$P(100,3) = 970{,}200$

So $C(100,3) = \dfrac{970{,}200}{3!} = 161{,}700$ possible collections of winners.

Example 3. Calculate $C(11,3)$ and $C(11,8)$. Do you notice anything about these two numbers? Can you give a reason why what you noticed must be true?

$C(11,3) = \dfrac{11!}{3!\cdot 8!} = \dfrac{11\cdot 10\cdot 9}{3\cdot 2\cdot 1} = 11\cdot 5\cdot 3 = 165$

$C(11,8) = \dfrac{11!}{8!\,3!} = \dfrac{11\cdot 10\cdot 9}{3\cdot 2\cdot 1} = 11\cdot 5\cdot 3 = 165$

They are equal. Choosing 3 things is the same as not choosing 8 things

Example 4. A bag contains three red marbles, three purple marbles, three green marbles and two yellow marbles (all of which are distinguishable from one another.)

(a) How many sets of four marbles are possible?

(b) How many sets of four are there such that each one is a different color?

(c) How many sets of four are there in which at least two are red?

(d) How many sets of four are there in which none are red, but at least one is green?

(a) $C(11,4) = \dfrac{11!}{4!\cdot 9!} = 330$ possible sets.

(b) Choose a red one: $C(3,1) = 3$
Choose a purple one: $C(3,1) = 3$
Choose a green one: $C(3,1) = 3$
Choose a yellow one: $C(2,1) = 2$

$3\cdot 3\cdot 3\cdot 2 = 54$ possible sets

(c) Alt. 1:
Choose two red ones: $C(3,2) = 3$
Choose two ~~nonred~~: $C(8,2) = 28$
Alt 2:
Choose 3 red: $C(3,3) = 1$
Choose 1 ...: $C(8,1) = 8$

$3\cdot 28 + 1\cdot 8 = 92$ sets

(d) Three alternatives: 1, 2, 3 Green

or

$C(8,4) = 70$ have no red.
$C(5,4) = 5$ have no green

65 possibilities.